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Explaining Cobb-Douglas with the New Mathematics of Inteduct

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Abstract

The Cobb-Douglas production function is used for around a hundred years and describes the macro economic reality very well. Though nobody doubts its validity, there is no rigorous derivation of it. Giving this derivation is done in this publication for the first time. We start with the most general version of a production function just assuming that there exists such function of an arbitrary number of variables (all possible input factors). We then make a Taylor expansion of this function still containing many variables. We then *average* over groups of variables via an inteduct. The groups can be e.g. labor and capital leading exactly the well-known version of the Cobb-Douglas production function. As we know how to derive this function, we also know the origin of the normally small deviations between Cobb-Douglas and the reality.

Keywords

Cobb-Douglas, Production Function, Taylor, Inteduct, Climate Change

1. Introduction

The Cobb-Douglas production function (Cobb & Douglas, 1928) states that the total production is proportional to a product of labor (with a non-integer exponent) and capital (also with a non-integer exponent) and maybe more production factors. All exponents must add up to one. For a simple version please see Equation (8) below. For an overview please see e.g. Wikipedia, 2023.

Cobb-Douglas has been used successfully for around hundred years. It coincides with reality fairly well. Therefore almost nobody (including the authors of this publication) doubts that Cobb-Douglas is *correct*. One criticism on Cobb-Douglas is that there is no rigorous theoretical foundation. It is more or less *assumed*.

The main point of this publication is to give a rigorous derivation of the Cobb-Douglas production function.

Though Cobb-Douglas is at first glance plausible, and it mirrors reality well, it is odd that it is a non-analytic function due to its non-integer exponents. From a purely mathematical point of view non-analytic functions are not strange. However, in fields which have to do with measurable results like economics and even (theoretical) physics any non-analytical function can be expressed with arbitrary accuracy by an analytic function. This leads to the sloppy statement that *every* function is analytic (Schulz, 2015).

If the production function is at least in arbitrary accuracy analytic, there should be a Taylor expansion of it. A Taylor expansion of a function will lead to a power series of its variables (with integer exponents). With this logic there must be a power series of the Cobb-Douglas production function. In chapter 3 we show (Equation (10) there) its explicit form. It contains with no surprise very many variables as the production of an economy or industry has a huge number of inputs.

In order to get the form of the Cobb-Douglas production function with a few variables (production factors) we group these waste number of variables by an average over a few groups via a fairly new tool called inteduct, Grabinski & Klinkova, 2023 (An inteduct is a generalized geometric mean in the same way as an integral is generalized sum).

From this we result in a rigorous derivation of the Cobb-Douglas production function. As stated, our derivation is based upon a Taylor expansion, see e.g. Bronshtein et al., 2007. In chapter 2 we will give a brief summary of Taylor expansions as far it is essential for chapter 3.

In chapter 4 we will draw conclusions. As we have *derived* the production function we can explicitly state which simplifications are in the Cobb-Douglas production function. Any difference between the standard production function and reality must originate in these simplifications. But this is left for future work.

2. Fitting with a Power Series

There is a theorem going back to Brook Taylor (see e.g. Bronshtein et al., 2007):

$$f(x)=\sum_{k=0}^{n}\frac{x^k}{k!}\cdot\frac{\partial^k f}{\partial x^k}(0)+\frac{x^{n+1}}{(n+1)!}\cdot\frac{\partial^{n+1} f}{\partial x^{n+1}}(\theta\cdot x) \tag{1}$$

where $0<\theta<1$ and $f(x)$ is a function $\mathbb{R}\mapsto\mathbb{R}$ being at least $n+1$ times differentiable. It is easy to see that Equation (1) will converge for $n\to\infty$. This leads to a more common form of Equation (1):

$$f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\cdot\frac{\partial^k f}{\partial x^k}(0) \tag{2}$$

Equation (2) states that any analytical function $f(x)$ can be expressed in a power series. This is commonly referred to as Taylor expansion. It is the basic for any kind of fit. In the literature one quite often finds using $f(x+a)$ instead of $f(x)$. People speak of an *expansion around* a. Please note that this is possible but superfluous. It seems to be necessary if someone tries to make a Taylor series for e.g. $\ln x$ as the logarithm (and all its derivatives) do not exist at $x=0$. However, using e.g. $\ln(1+x)$ instead of $\ln x$ will eliminate the problem.

As an example consider

$$\frac{1}{1+x}=\sum_{k=0}^{\infty}\frac{x^k}{k!}\cdot(-1)^k \tag{3}$$

In many applications one uses an expansion up to a certain order. The second order expansion of Equation (3) is sometimes displayed as

$$\frac{1}{1+x}\approx 1-x+x^2 \tag{4}$$

It is clear what is meant by it. However, the ≈ sign is not well defined and therefore mathematically tricky (Using a final term as in Equation (1) makes it exact but inconvenient to handle). A writing such as

$$\frac{1}{1+x}=1-x+x^2+O\left(x^3\right) \tag{5}$$

is much more convenient. The $O\left(x^3\right)$ term means that there are following terms of at least third power in *x*. As stated, a Taylor expansion is only possible for analytic functions. From the point of pure mathematics this is a real limitation. In any applied field functions display something which can be measured from reality. Any at least piecewise analytic function can be approximated within arbitrary accuracy by a completely analytic function. In this sense a discontinuity is e.g. nothing but an arbitrarily big slope. Therefore in applied fields like economics or even in (theoretical) physics one may formulate the following statement: *Every* function can be expressed by a Taylor series.

Needless to say that a Taylor series can be generalized to an arbitrary amount of variables. Just consider a function $f(x,y)$. To keep it simple consider the Taylor expansion of *f* up to second order:

$$f(x,y)=a_{00}+a_{10}\cdot x+a_{01}\cdot y+a_{20}\cdot x^2+a_{02}\cdot y^2+a_{11}\cdot xy+O\left(x^3,y^3,x^2y,xy^2\right) \tag{6}$$

This is already a quite clumsy formula. It is possible to write Equation (6) in the form of Equation (2) even for an arbitrary number of variables. For completeness we will also give the a_{ij}:

$$\begin{aligned} &a_{00}=f(0,0),\ a_{10}=\frac{\partial f}{\partial x}(0,0),\ a_{01}=\frac{\partial f}{\partial y}(0,0)\\ &a_{20}=\frac{\partial^2 f}{\partial x^2}(0,0),\ a_{02}=\frac{\partial^2 f}{\partial y^2}(0,0),\ a_{11}=\frac{\partial^2 f}{\partial x\partial y}(0,0) \end{aligned} \tag{7}$$

Most expansions will go only to linear order. Especially in economics there are standard procedures to check whether higher (quadratic) orders are important.

As it is easily possible to check whether the terms $a_{20} \cdot x^2$ or $a_{02} \cdot y^2$ of Equation (6) are necessary to consider, it is next to impossible to prove the necessity of $a_{11} \cdot xy$ in Equation (6). Cross terms like e.g. $a_{11} \cdot xy$ are commonly neglected (or better ignored). This is especially true for complex models where *all* variables are far from known. As an infamous example see e.g. the economic consequences of climate change (Nordhaus, 1979). Here it is calculated that a certain temperature raise will lead to an economic optimum between the current investment for reducing CO_2 and the cost to counter act the changed climate. As the true variables are even not known from physics and the areas of stability are far from being understood, a cost optimum cannot be calculated for mathematical reasons. But this would be a topic for a separate paper.

Please note that the authors of this paper do not mean that counter measures for climate change are superfluous. It is just the calculation in Nordhaus (1979), which is faulty. Finally, Nordhaus (1979) led to the acceptance of goals in climate summits. This deserves credit in its own right.

3. Deriving Cobb-Douglas from Taylor

We have just shown that for practical purposes *every* function can be expressed in a Taylor series. In this context it appears to be pretty odd that there is a so-called Cobb-Douglas production function:

$$p(l,c) = a \cdot l^{1-\alpha} \cdot c^{\alpha} \tag{8}$$

Here p refers to the total amount of goods produced in a country or industry (normally per year) measured in a currency unit. l is the necessary amount of labor for it, normally also measured in a currency unit, and c is the necessary capital to be invested in machines and other equipment. The constant a is commonly referred total factor productivity.

It is no surprise that there exists a function $p(l,c)$ as labor and capital are the main ingredients for producing goods. (The same is true for services) However, Equation (8) is non-analytic. It is not a power series. Though it is a very practical formula, $p(l,c)$ is not a Taylor series and cannot be expressed in one. This is in contrast to main result of the previous chapter. To solve this conundrum is the core of this publication.

Before coming back to this conundrum, we will briefly discuss the well-known Cobb-Douglas production function of Equation (8). Textbooks or even Wikipedia, 2023 give perfect overviews. It is clear that the exponents in Equation (8) must add to one. It is a consequence of scale invariance or dimension analysis. Though a production function has been used earlier, Cobb & Douglas (1928) used it to estimate $\alpha \approx 0.75$ by making a least square fit. Equation (8) has been used in many other situations with almost perfect agreement to reality. Li (2023) used it successfully for a climate change model of rice production in Japan. The production function can be generalized to more than two production factors (here l and c). Of course, the sum of the exponents must always be one.

There is no doubt that Equation (8) is correct. The main criticism is that there is no *derivation* of Equation (8). And as argued here, it appears to be pretty odd that the production function is not an analytical function. Even in physics such non-analytic behavior is pretty rare. It appears generally only at critical points, see e.g. Grabinski, 1990. Simplified, this is a point where one description by a Taylor series changes to another region with a different Taylor series. This is for sure not given here. It can be (and most likely is) an issue in e.g. climate change models like Nordhaus, 1979.

One step for a solution is that l and c cannot be the total investment in labor or capital, respectively. It must be the *necessary* investment to produce the goods of an economy or industry. There are a very big number of units of labor (e.g. turning a certain screw) and investments in capital (for e.g. the screwdriver). So we have a very big number of variables:

$$x_1, x_2, \cdots, x_M, x_{M+1}, \cdots, x_N \tag{9}$$

M and $N - M$ are $\gg 1$. One of the x_j will refer to the above mentioned turning of a screw and another will be the investment into a screwdriver. As all x_j are necessary the production function, $p(x_1, x_2, \cdots, x_N)$ has a Taylor series of the form:

$$p(x_1, x_2, \cdots, x_N) = \tilde{a} \cdot \prod_{j=1}^{N} x_j + O\left(x_i^{N+1}\right) \tag{10}$$

$$\tilde{a} = \frac{\partial^N p}{\prod_{j=1}^{N} \partial x_j}(0, 0, \cdots, 0) \tag{11}$$

Equation (10) is the Taylor series of $p(x_1, x_2, \cdots, x_N)$ to lowest order. It is clear that there is no constant term as nothing will be produced without labor or capital. As all x_j are necessary, the Taylor series must start with a product (cross term) of *all* variables. Equation (10) is for sure correct but almost useless as the variables $x_1, x_2, \cdots, x_N$ are far from being known. In Equation (9) the variables $x_1, x_2, \cdots, x_M, x_{M+1}, \cdots, x_N$ are deliberately grouped into $x_1, x_2, \cdots, x_M$ and $x_{M+1}, \cdots, x_N$, respectively. The first group refers to variables which refer to labor the second to capital. In order to get something like Equation (8) there should be some kind of *averaging*.

The $x_1, x_2, \cdots, x_N$ build an N-dimensional space. One may order these N variables in an interval $[0,1]$. As M and $N - M$ are $\gg 1$ one may assume that the positions of the x_j take (almost) all rational numbers in $[0,1]$. To take an average over these variables of which all are necessary one may use an inteduct as defined by Grabinski & Klinkova, 2023.

The inteduct is defined for a function $g(x)$ within an interval $x \in [a,b]$ and $g(x) > 0 \quad \forall x \in [a,b]$ as

$$\prod_{a\ b}^{x} g(x) \equiv \lim_{n \to \infty} \left(\prod_{k=0}^{n} g\left(a + \frac{k}{n} \cdot (b - a) \right) \right)^{\frac{1}{n+1}} \tag{12}$$

It is a generalized geometric mean. As argued in Grabinski & Klinkova, 2023, it is a perfect average for countable infinite number of quantities where each is important. In Grabinski & Klinkova, 2023 it is shown how to *split* an inteduct (Equation (14) there). Adopted to our situation we may write:

$$\prod_{0\,1}^{x} g(x) = \left(\prod_{0\,\alpha}^{x} g(x)\right)^{\alpha} \cdot \left(\prod_{\alpha\,1}^{x} g(x)\right)^{1-\alpha} \tag{13}$$

Let us assume the variables $x_1, x_2, \cdots, x_M$ are in the interval $[0, \alpha]$ and $x_{M+1}, \cdots, x_N$ are in $[\alpha, 1]$ (In this sense we have $M/N = \alpha$). As the variables $x_1, x_2, \cdots, x_M$ correspond to labor and $x_{M+1}, \cdots, x_N$ to capital one may take the following definitions:

$$\prod_{0\,\alpha}^{x} g(x) \equiv \tilde{l} \text{ and } \prod_{\alpha\,1}^{x} g(x) \equiv \tilde{c} \tag{14}$$

From Equation (13) with (14) we find that the average of the product of the x_j can be expressed in Equation (10) as:

$$p(x_1, x_2, \cdots, x_N) = \tilde{a} \cdot \tilde{l}^{\alpha} \cdot \tilde{c}^{1-\alpha} + O\left(x_i^{N+1}\right) \tag{15}$$

Though the $\tilde{l}$ and $\tilde{c}$ in Equation (15) are useful averages of the corresponding x_j, $\tilde{l}$ and $\tilde{c}$ are not identical to l and c of Equation (8). However, Equation (15) can be written as:

$$p(x_1, x_2, \cdots, x_N) = \tilde{a} \cdot \left(\frac{l}{l/\tilde{l}}\right)^{\alpha} \cdot \left(\frac{c}{c/\tilde{c}}\right)^{1-\alpha} + O\left(x_i^{N+1}\right) \tag{16}$$

With the definition

$$\tilde{a} \cdot \left(\frac{\tilde{l}}{l}\right)^{\alpha} \left(\frac{\tilde{c}}{c}\right)^{1-\alpha} \equiv a \tag{17}$$

Equation (16) is almost identical to Equation (8). Comparing it to Equation (10) one sees that the production function is nothing but lowest order expansion with *averaged* variables l and c as defined in Equation (14).

Please note that our derivation is easily generalized to an arbitrary number of production factors. It just means that a split in Equation (13) is extended to more than two factors.

Please note that it is not necessary to have a countable infinite number of variables x_j. Equation (12) without the limit $n \to \infty$ is nothing but generalized geometric mean. Without the limit $n \to \infty$ Equation (12) looks identical as long as $M/(M+1) \approx 1$ or $(N-M)/(N-M+1) \approx 1$, respectively, can be assumed.

4. Conclusions and Further Work

We have shown that the Cobb-Douglas production function can be rigorously derived from a Taylor expansion if one assumes averaged variables via inteduct. This solves a great puzzle as the main criticism of the Cobb-Douglas production function is that there is no rigorous theoretical justification for it and all the more no strict derivation, see e.g. Wikipedia, 2023.

There are attempts to explain Cobb-Douglas. Far from complete they are e.g. Houthakker, 1955, and Jones, 2004, and Simon & Levy, 1963, and Walsh, 2017. All have in common that they at most make Cobb-Douglas more plausible. They are far apart from the only assumption made here: The *existence* of a production function of an arbitrary number of variables.

Especially in Jones, 2004, other conditions like a Pareto distribution have been assumed. This is not implausible but on the other hand not proven either. Furthermore, as in almost all economic models a *stable* equilibrium is assumed. Though the word *stable* is almost never mentioned, it is essential. That stability is not trivial has been shown in Schädler & Grabinski, 2015 for the NAIRU (nonaccelerating inflation rate of unemployment) model which goes back to Tobin, 1980.

Though the Cobb-Douglas production function is in very good accordance with reality, there are (small) deviations. As we present a clear-cut derivation, we are able to at least give hints, where these deviations might come from. Our Equation (10) is for sure exact. Coming to Equation (8) or (15) some simplifications has been made. In these simplifications the deviations must origin.

The first simplification is that Cobb-Douglas uses the lowest non-trivial order only. In principle there are higher order terms where one or more x_j appear in higher orders. At least for the authors it appears to be unplausible that this will lead to a measurable effect. As the x_j are elementary and therefore arbitrarily small, the lowest order should be sufficient.

The second simplification is the use of *averages* via inteduct. The authors believe that the resulting error is pretty small though it is next to impossible to prove it. Probable an adjusted total factor productivity *a* will compensate for this error. A proof would imply to measure the difference between Equation (15) and (10) which is next to impossible as Equation (10) cannot be handled because of its waste number of variables which are also unknown in particular.

More technically speaking, the averaging via inteduct led to an a defined in Equation (17). Our second simplification is nothing but the assumption that a is not a function of *l* or *c*, respectively. As $\tilde{l}$ and $\tilde{c}$ are (essentially) proportional to *l* and *c*, respectively, it appears to be a realistic assumption.

The third simplification is that only the *necessary* capital and labor are considered in the x_j. Probably everybody using Cobb-Douglas will agree to this triviality. However, one only measures *l* and *c* which corresponds to total labor and capital costs. They are easy to measure but it remains unclear whether they are necessary. Especially in socialist economies (with no unemployment) it has been joked that some people were digging to create a trench and others immediately closed this very same trench. All people digging might have shown an extreme productivity or efficiency but it was useless. Especially when such *digging* is subtler, it will be very hard to discover.

The fourth simplification is that only two production factors (labor and capital) have been used here. Of course there are Cobb-Douglas production functions taking into account more than two factors. And such considerations are in accordance with our derivation. However, if these factors include things like envy, pride, and the like, it might become very tricky. For sure such factors will influence the total output. But they are in general no mathematical objects which can be added or multiplied even if (monetary) values are assigned. For more detail please see Lunkenheimer et al., 2022.

From the just mentioned four points it is pretty clear how a future work could look like. For any essential deviation between the Cobb-Douglas production function and measurements of reality it should be checked whether one of the four points above is the reason behind it.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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